Quantifying Resilience of Ocean Circulation in Simple Box Models

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Mathematics of Climate Seminar

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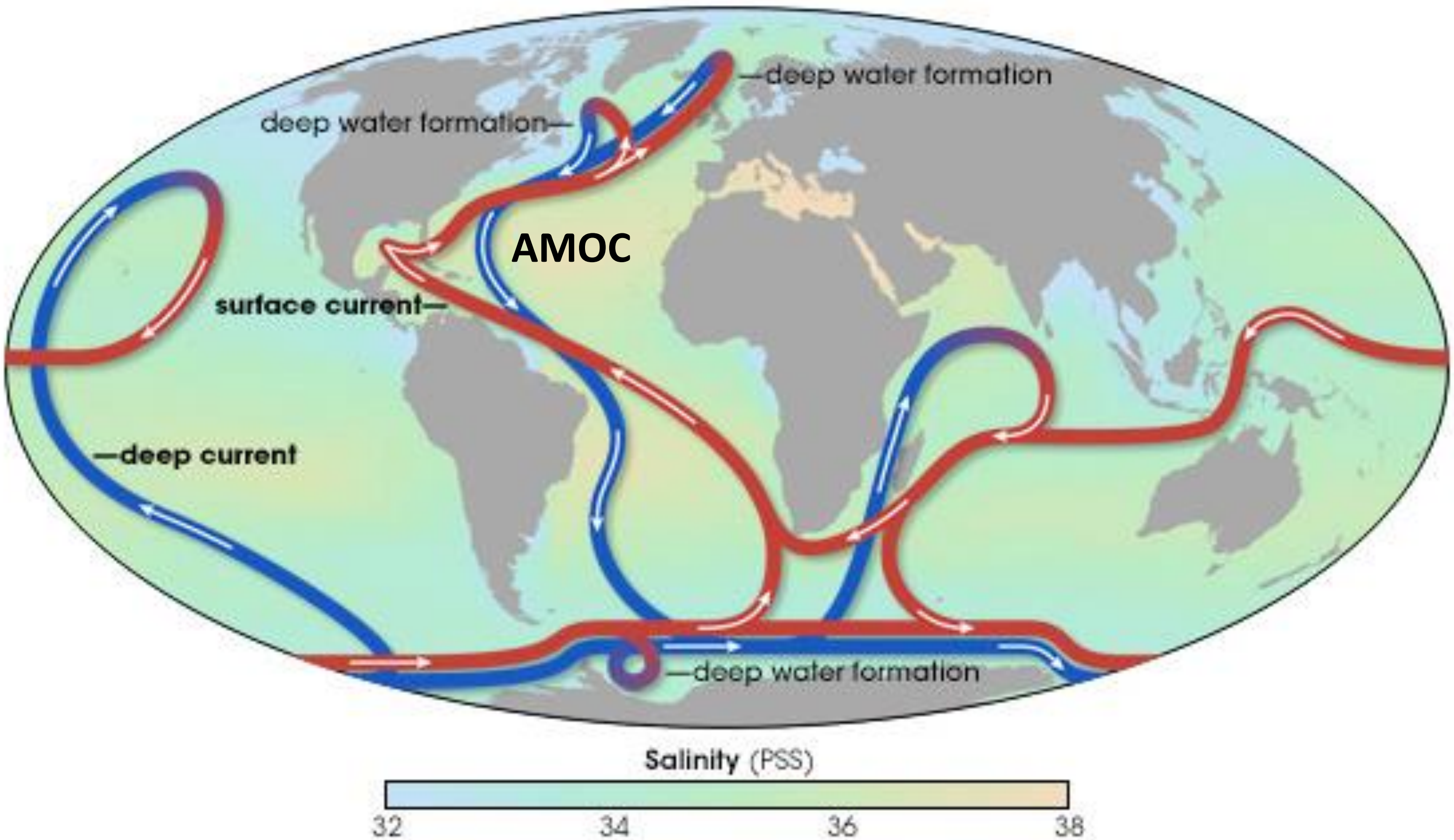
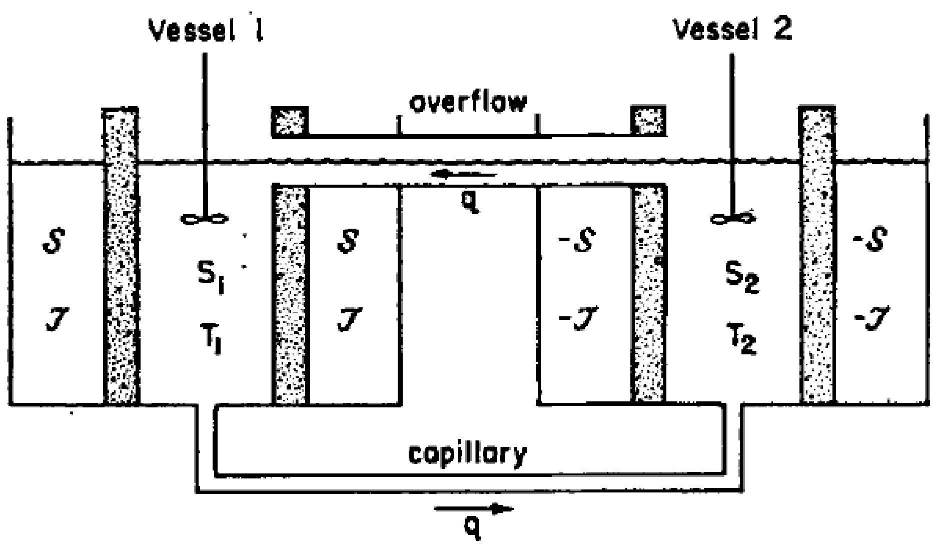
Oceans: a key player in climate dynamics



http://earthobservatory.nasa.gov/Features/Paleoclimatology_Evidence/paleoclimatology_evidence_2.php

Q: How resilient is the AMOC to disruptions?

[**resilience:**
the capacity of a system to absorb disturbance
and maintain its basic structure and function]

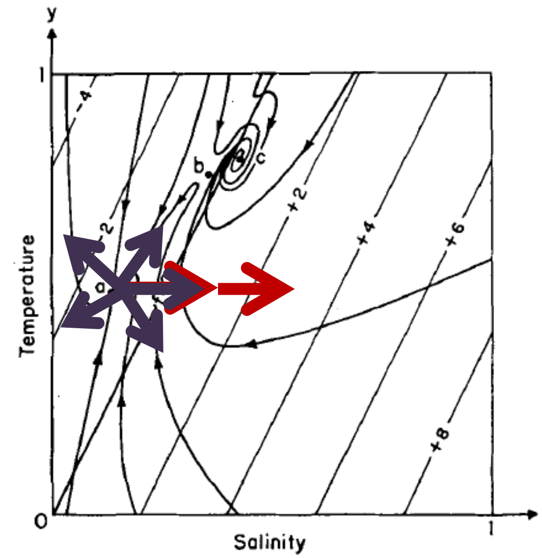


(Stommel 1961)

Possible Disruptions

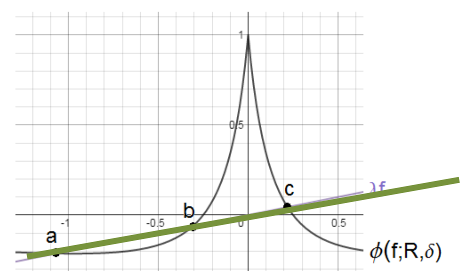
To State Variables:

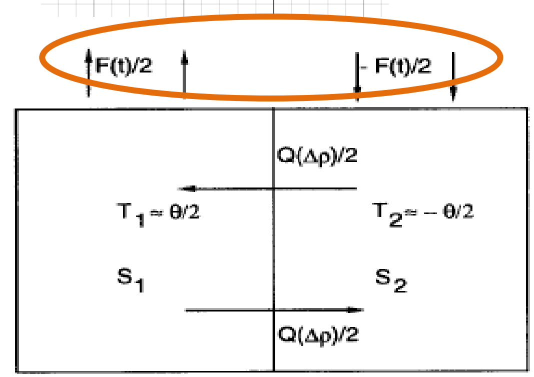
1. Repeated salinity “kicks”
2. Repeated “kicks” in any direction



To Parameters:

1. changes to parameter λ
2. changes to salinity forcing (Cessi’s adaptation)





Possible Disruptions

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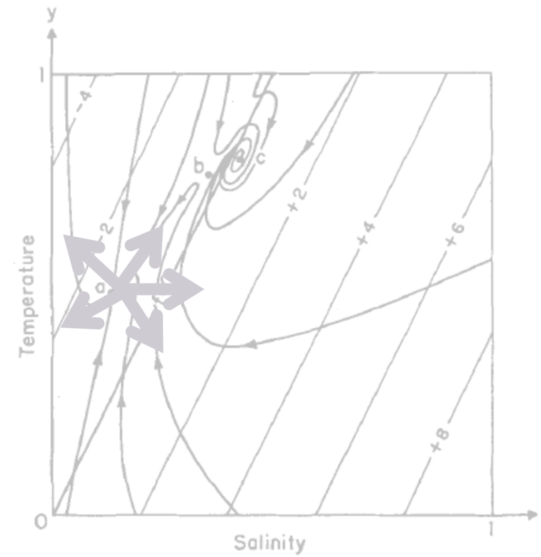

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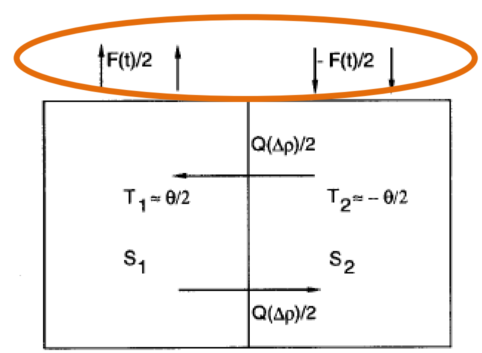
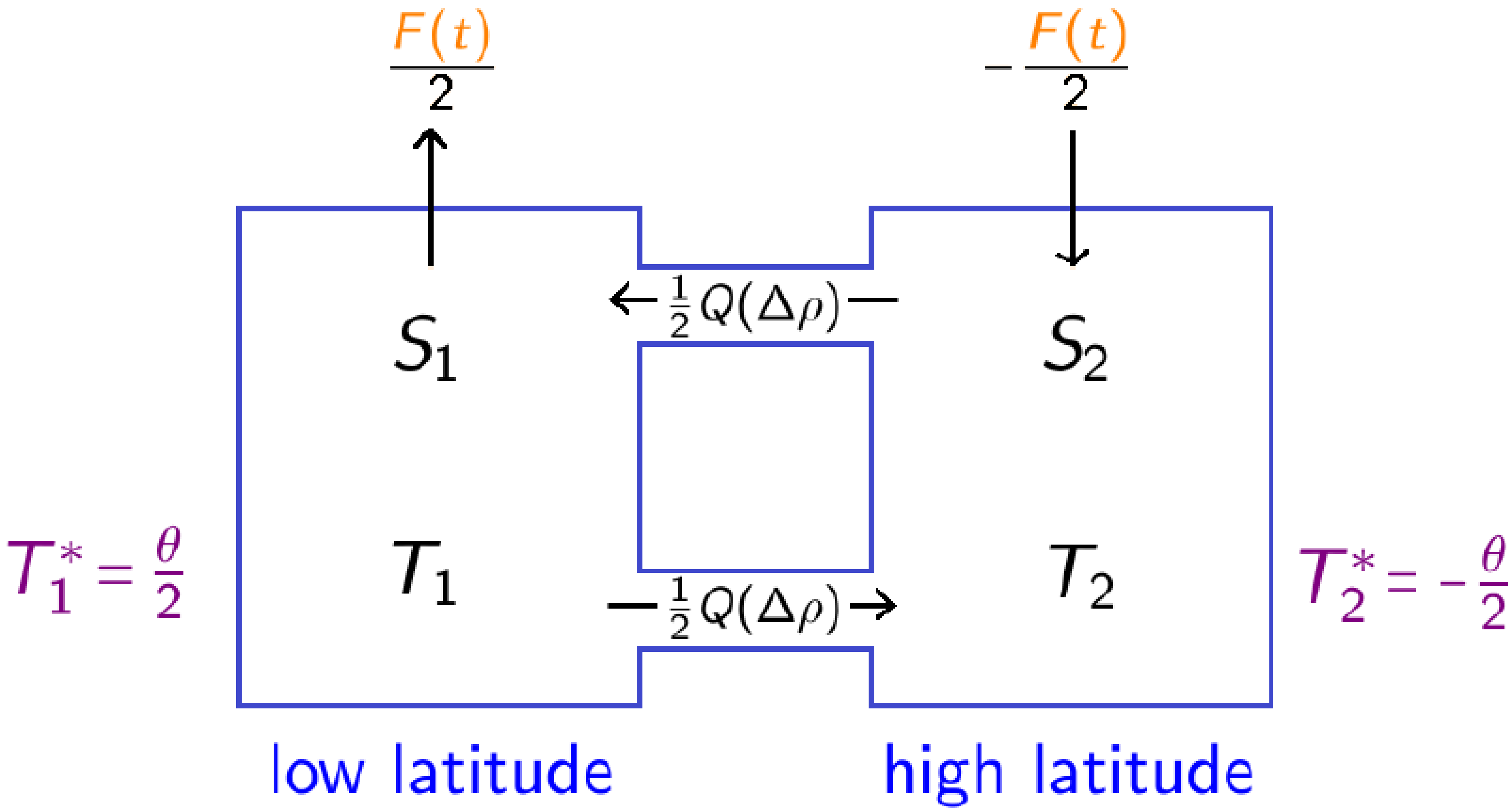


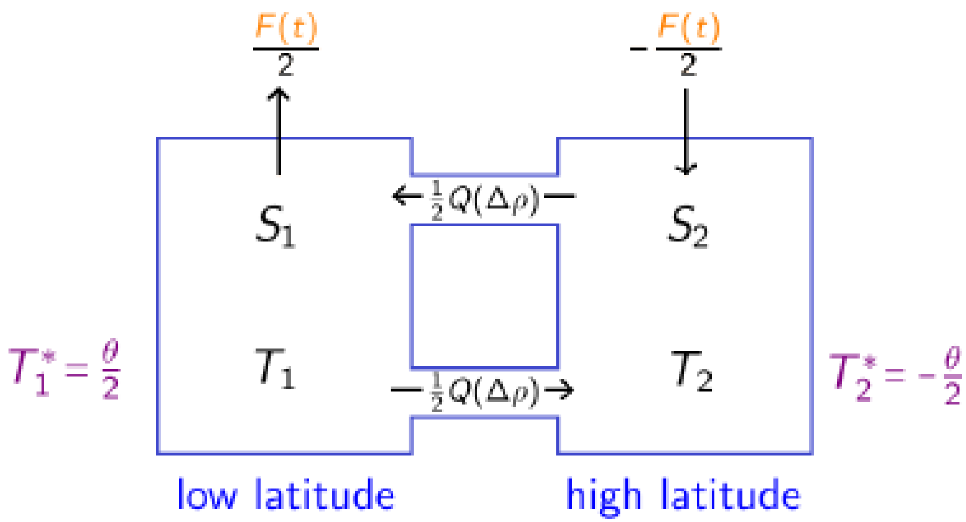
Cessi's Adaptation of Stommel



$$\Delta\rho = \alpha_S(S_1 - S_2) - \alpha_T(T_1 - T_2)$$

$$Q(\Delta\rho) = \frac{1}{t_d} + \frac{q(\Delta\rho)^2}{V}$$

Cessi's Original Set of Equations



$$T_1' = -\frac{1}{t_r}\left(T_1 - \frac{\theta}{2}\right) - \frac{1}{2}Q(\Delta\rho)(T_1 - T_2)$$

$$T_2' = -\frac{1}{t_r}\left(T_1 + \frac{\theta}{2}\right) - \frac{1}{2}Q(\Delta\rho)(T_2 - T_1)$$

$$S_1' = \frac{F(t)}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_1 - S_2)$$

$$S_2' = -\frac{F(t)}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_2 - S_1)$$

Four to Two Dimensions

$$T_1' = -\frac{1}{t_r}\left(T_1 - \frac{\theta}{2}\right) - \frac{1}{2}Q(\Delta\rho)(T_1 - T_2) \qquad S_1' = \frac{F(t)}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_1 - S_2)$$

$$T_2' = -\frac{1}{t_r}\left(T_1 + \frac{\theta}{2}\right) - \frac{1}{2}Q(\Delta\rho)(T_2 - T_1) \qquad S_2' = -\frac{F(t)}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_2 - S_1)$$

$$\Delta T \equiv T_1 - T_2; \qquad \Delta S \equiv S_1 - S_2$$

$$\Delta T' = -\frac{1}{t_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$

$$\Delta S' = \frac{F(t)}{H}S_0 - Q(\Delta\rho)\Delta S$$

Non-dimensionalization

$$\Delta T' = -\frac{1}{t_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$

$$\Delta S' = \frac{F(t)}{H} S_0 - Q(\Delta\rho)\Delta S$$

$$\Delta\rho = \alpha_S(S_1 - S_2) - \alpha_T(T_1 - T_2)$$

$$Q(\Delta\rho) = \frac{1}{t_d} + \frac{q(\Delta\rho)^2}{V}$$

$$x \equiv \frac{\Delta T}{\theta}, \qquad y \equiv \frac{\alpha_S \Delta S}{\alpha_T \theta}, \qquad t \equiv t_d t'$$

$$\alpha \equiv \frac{t_d}{t_r}, \qquad p(t) \equiv \frac{\alpha_S S_0 t_d}{\alpha_T \theta H} F(t), \qquad \mu^2 \equiv \frac{q t_d (\alpha_T \theta)^2}{V}$$

$$x' = -\alpha(x - 1) - x\left[1 + \mu^2 (x - y)^2\right]$$

$$y' = p(t) - y\left[1 + \mu^2 (x - y)^2\right]$$

Cessi's Final Dimension Reduction

Non-dimensionalized equations

$$x' = -\alpha(x-1) - x\left[1 + \mu^2(x-y)^2\right]$$
$$y' = p(t) - y\left[1 + \mu^2(x-y)^2\right]$$

Since $\alpha \equiv \frac{t_d}{t_r}$ is very large,

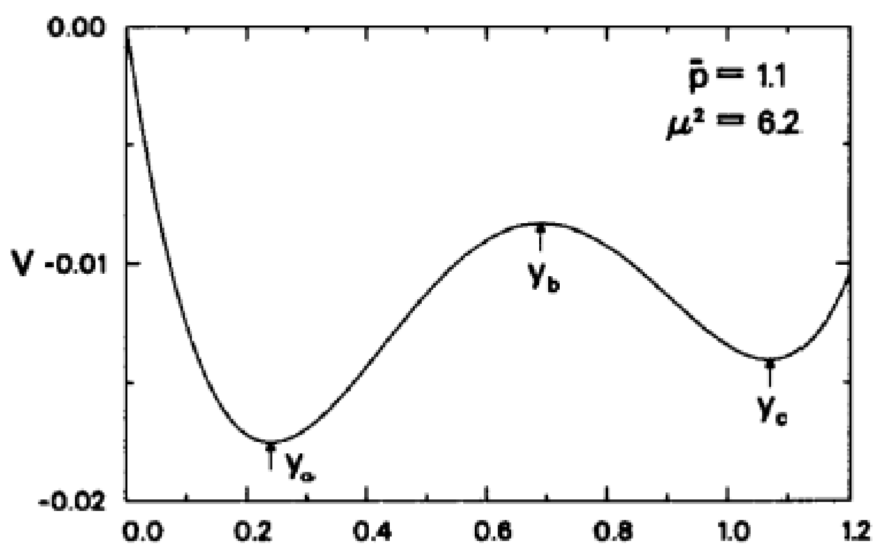
$$x = 1 + \mathcal{O}(\alpha^{-1})$$
$$y' = p(t) - y\left[1 + \mu^2(1-y)^2\right] + \mathcal{O}(\alpha^{-1})$$

$$x \approx 1$$
$$y' \approx p(t) - y\left[1 + \mu^2(1-y)^2\right]$$

The Unperturbed System

$$y' = p(t) - y\left[1 + \mu^2(1-y)^2\right]$$
$$= \bar{p} + \hat{p}(t) - y\left[1 + \mu^2(1-y)^2\right]$$

$$\hat{p}(t) = 0 \implies y' = -\frac{\partial}{\partial y}\Big[\overbrace{\mu^2\left(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right) + \frac{y^2}{2} - \bar{p}y}^{V(y)}\Big]$$



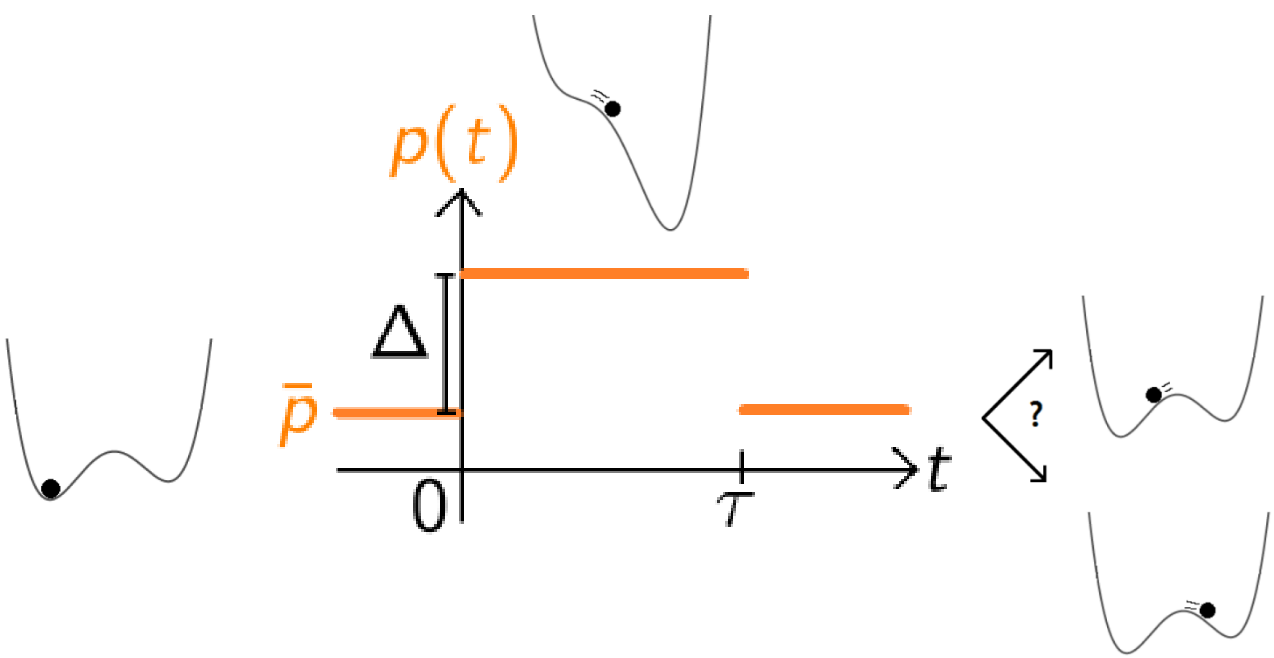
(Cessi 1994)

https://www.desmos.com/calculator/uyfur3oshz

The Perturbed System

$$y' = \bar{p} + \hat{p}(t) - y\left[1 + \mu^2(1-y)^2\right]$$

$$\text{Let } \hat{p}(t) = \begin{cases} 0 & t \le 0 \\ \Delta & 0 \le t \le \tau \\ 0 & t > \tau \end{cases}$$



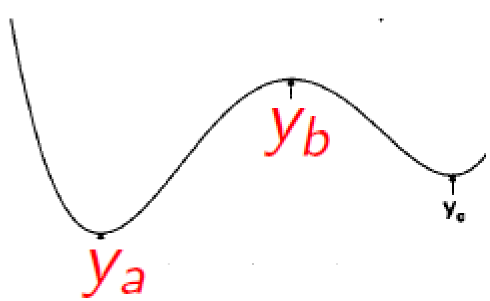
Quantifying Resilience

For $\Delta > \Delta_0$, how long can the system tolerate $p = \bar{p} + \Delta$ and still recover to y_a?

$$\frac{dy}{dt} = \bar{p} + \Delta - y\left[1 + \mu^2(1-y)^2\right]$$

$$\frac{dy}{\bar{p} + \Delta - y\left[1 + \mu^2(1-y)^2\right]} = dt$$

$$\int_{y_a}^{y_b} \frac{dy}{\bar{p} + \Delta - y\left[1 + \mu^2(1-y)^2\right]} = \int_0^{\tau} dt = \tau$$



Quantifying Resilience

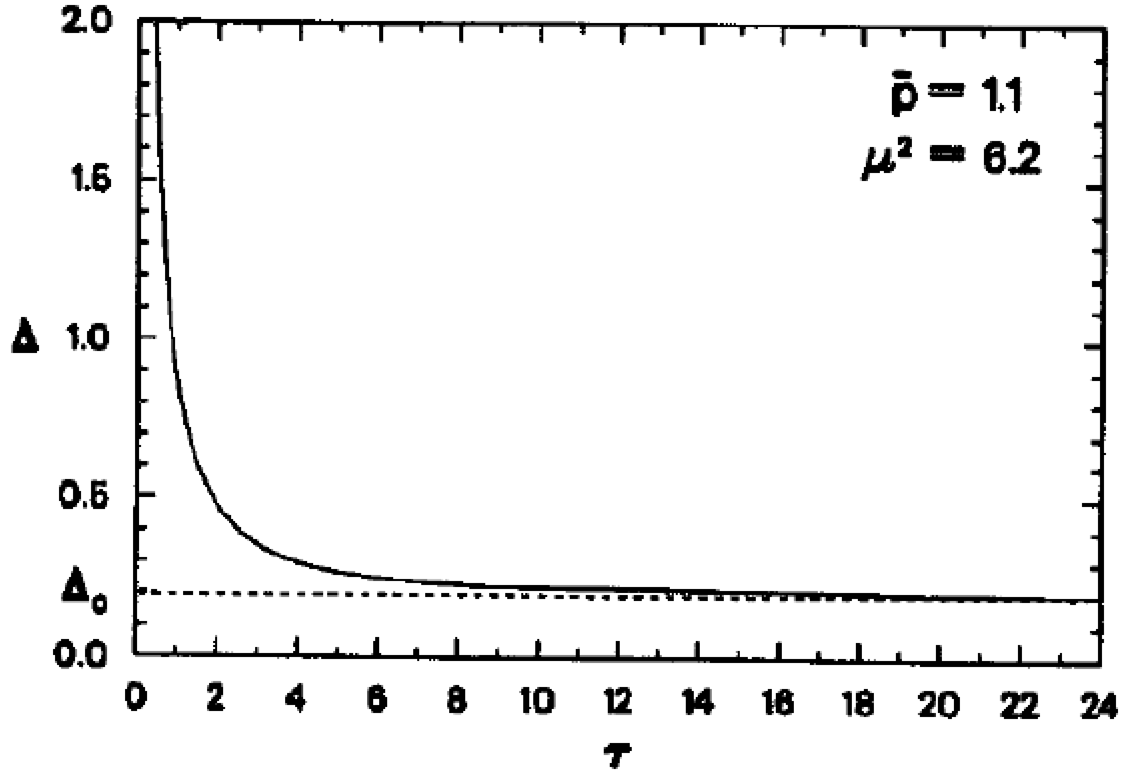


FIG. 3. The minimum amplitude of a perturbation, as a function of its duration, that will shift the system from the globally stable equilibrium y_a of Fig. 2 to the metastable state, y_c. The perturbation must exceed a critical amplitude, Δ_0, in order to displace the system from y_a, even if applied for an infinite time.

(Cessi 1994)

Cessi's Calculation

- 1,000 years $\longrightarrow \tau = 4.6$
- Critical value of Δ for $\tau = 4.6$ is $\Delta \approx 0.3$
- $\Delta_0 = 0.2$ corresponds to freshwater flux of $0.4\ m\ yr^{-1}$
- Max meltwater flux preceeding Younger Dryas was $0.5\ m\ yr^{-1}$

"Close"

Questions

- Can Cessi's method be extended to higher dimensions?
- Can we quantify resilience to continuous parameter changes?
- Yours?

References

Cessi, Paola. 1994. A simple box model of stochastically forced thermohaline flow. *Journal of Physical Oceanography* v. 24, 1911-1920

Stommel, Henry. 1961. Thermohaline convection with two stable regimes of flow, Tellus XIII, 2 pp. 224-230